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# An Equivalence Result in Linear-Quadratic Theory

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## Abstract

We consider the zero-endpoint infinite-horizon LQ problem. We show that the existence of an optimal policy in the class of feedback controls is a sufficient condition for the existence of a stabilizing solution to the algebraic Riccati equation. This result is shown without assuming positive definiteness of the state weighting matrix. The feedback formulation of the optimization problem is natural in the context of differential games and we provide a characterization of feedback Nash equilibria both in a deterministic and stochastic context.

**Keywords:** LQ theory, algebraic Riccati equations, LQG theory, differential games.

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\*This author performed this research while at Tilburg University

# 1 Introduction

The indefinite, regular, zero-endpoint, infinite-horizon LQ (IRZILQ) problem is the problem of finding a control function  $u \in U_s(x_0)$  for each  $x_0 \in \mathbb{R}^n$  that minimizes the cost functional

$$J(x_0, u) := \int_0^\infty (x^\top Q x + u^\top R u) dt, \quad (1)$$

with  $Q = Q^\top$ ,  $R > 0$ , and where the state variable  $x$  is the solution of  $\dot{x} = Ax + Bu$ ,  $x(0) = x_0$ . Here the class of control functions  $U_s(x_0)$  is defined by:

$$U_s(x_0) = \left\{ u \in L_{2,\text{loc}} \mid J(x_0, u) \text{ exists in } \mathbb{R} \cup \{-\infty, \infty\}, \lim_{t \rightarrow \infty} x(t) = 0 \right\}$$

(see e.g. Willems, 1971, Molinari, 1977, or Trentelman & Willems, 1991). In addition to this class of control functions we also consider the set of linear, internally stabilizing, time-invariant feedback matrices, i.e.

$$\mathcal{F} := \{F \mid A + BF \text{ is stable}\}.$$

A state feedback control function corresponding to a feedback matrix  $F$  and an initial state  $x_0$  is denoted by  $u^{\text{FB}}(x_0, F)$ . The following statements are relevant for the IRZILQ problem:

- (i)  $\forall x_0 \exists \hat{u} \in U_s(x_0) \forall u \in U_s(x_0) \quad J(x_0, \hat{u}) \leq J(x_0, u);$
- (ii)  $\exists \hat{F} \in \mathcal{F} \forall x_0 \forall u \in U_s(x_0) \quad J(x_0, u^{\text{FB}}(x_0, \hat{F})) \leq J(x_0, u);$
- (iii)  $\exists \hat{F} \in \mathcal{F} \forall x_0 \forall F \in \mathcal{F} \quad J(x_0, u^{\text{FB}}(x_0, \hat{F})) \leq J(x_0, u^{\text{FB}}(x_0, F));$
- (iv)  $\Delta$  is positive definite where  $\Delta$  denotes the difference between the largest and smallest real symmetric solution of the algebraic Riccati equation (3) below;
- (v) The algebraic Riccati equation (3) has a stabilizing solution.

It is immediately clear that (ii)  $\Rightarrow$  (i),(iii). Willems (1971, Theorems 5 and 7) (see also Trentelman & Willems, 1991, Theorem 8.8.2) showed that (i)  $\Leftrightarrow$  (iv)  $\Rightarrow$  (ii) and (iv)  $\Leftrightarrow$  (v) under the assumption that  $(A, B)$  is controllable and that the algebraic Riccati equation has a real symmetric solution. In the present paper we shall prove the equivalence (iii)  $\Leftrightarrow$  (v) in Section

3. This will be done without assuming controllability. The implication (v)  $\Rightarrow$  (iii) follows from a simple completion of the squares. Clearly, if the system is controllable, this relation also follows from the work of Willems via (v)  $\Rightarrow$  (iv)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii). The main contribution of the present paper is the converse implication, i.e. (iii)  $\Rightarrow$  (v). In fact this can be formulated as a parametric optimization problem and we will use a matrix differentiation argument to solve it. If the system is controllable, this result implies that all the statements (i), ..., (v) are equivalent. It is well-known from the certainty equivalence principle that the algebraic Riccati equation also appears in LQG optimal control theory. Therefore, it is to be expected that the equivalence result (iii)  $\Leftrightarrow$  (v) can be translated in a stochastic context. This is done in Section 4. More precisely, in this section we consider a sixth statement in addition to the statements (i), ..., (v) and show that this statement is also equivalent to statement (v).

We were motivated to study this problem by studying feedback Nash equilibria for infinite-horizon LQ differential games (Başar and Olsder, 1999). The equivalence (iii)  $\Leftrightarrow$  (v) leads straightforwardly to a characterization of feedback Nash equilibria in terms of stabilizing solutions of a set of coupled algebraic Riccati equations. In such an equilibrium, the strategy space of each player is restricted to linear time-invariant state feedback matrices, and furthermore, the resulting closed-loop system is required to be stable. A precise discussion is given in Section 5.1. The stochastic interpretation studied in Section 4 is generalized in Section 5.2 to a stochastic differential game setting. In this final section we define the concept of a stochastic variance-independent feedback Nash equilibrium.

## 2 Preliminaries

With a small abuse of notation we shall write  $J(x_0, F) := J(x_0, u^{\text{FB}}(x_0, F))$ . Clearly, for each  $F \in \mathcal{F}$  we have  $J(x_0, F) = x_0^\top \varphi(F) x_0$  with  $\varphi : \mathcal{F} \rightarrow \mathbb{R}^{n \times n}$  defined by  $\varphi : F \mapsto P$  where  $P$  is the unique solution of the Lyapunov equation

$$(A + BF)^\top P + P(A + BF) = -(Q + F^\top RF). \quad (2)$$

The algebraic Riccati equation (ARE) corresponding to our problem is given by

$$Q + A^\top X + XA - XBR^{-1}B^\top X = 0. \quad (3)$$

A solution  $X$  of this equation is called *stabilizing* if the matrix  $A - BR^{-1}B^\top X$  is stable. It is well-known (see e.g. Lancaster & Rodman, 1995, Proposition 7.9.2) that such a solution, if it exists, is unique.

If  $\mathcal{X}$  and  $\mathcal{Y}$  are finite dimensional vector spaces and  $\mathcal{D}$  is an open subset of  $\mathcal{X}$ , we denote the *derivative* of a differentiable map  $T : \mathcal{D} \rightarrow \mathcal{Y}$  by  $\partial T$  and the *differential* of  $T$  at  $x \in \mathcal{D}$  in the direction  $h$  by  $\delta T(x; h)$ . We have  $\delta T(x; h) = \partial T(x)h$  (see e.g. Luenberger, 1969, Chapter 7). Partial derivatives and differentials are denoted by  $\partial_i$  and  $\delta_i$  where the index refers to the corresponding argument.

### 3 Main Result

If the stabilizing solution  $X$  of the ARE exists, it follows from a standard completion of the squares (see e.g. Willems, 1971, Lemma 6) that

$$J(x_0, F) = x_0^\top X x_0 + \int_0^\infty x^\top (F - \hat{F})^\top R (F - \hat{F}) x dt$$

where  $\hat{F} := -R^{-1}B^\top X$ . This expression shows that  $J$  is minimized at  $\hat{F}$  for each initial state  $x_0$ . Of course, if the system is controllable, this fact has already been established by Willems (1971) as noted in the introduction. The next theorem states that the converse statement is also true. Its proof is based on a variational argument. Grabowski considered in (1993) the regular positive definite infinite-horizon LQ problem as a parametric optimization problem. Working in an infinite dimensional context he showed, under a detectability assumption, that the corresponding ARE has a stabilizing solution using a policy iteration argument. Here we do not assume detectability nor positive definiteness of the state weighting matrix.

**Theorem 3.1** *Consider the system  $\dot{x} = Ax + Bu$  with  $(A, B)$  stabilizable,  $u = Fx$ ; and the cost functional  $J(x_0, F) = x_0^\top \varphi(F) x_0$  with  $\varphi$  defined as in Section 2. If  $\hat{F} \in \mathcal{F}$  is a minimum*

for  $J$  for each  $x_0 \in \mathbb{R}^n$ , then  $X := \varphi(\hat{F})$  is the stabilizing solution of the ARE.

**Proof** First note that the set  $\mathcal{F}$  is a nonempty open set. Secondly note that the smoothness of the coefficients in a Lyapunov equation is preserved by the solution of this equation (see e.g. Lancaster & Rodman, 1995, Section 5.4), which implies that  $J$  is differentiable with respect to  $F$ . Now, let  $\hat{F} \in \mathcal{F}$  be a minimum of  $J$  for each  $x_0$ . Then (see e.g. Luenberger, 1969, Section 7.4, Theorem 1)  $\delta_2 J(x_0, \hat{F}; \Delta F) = 0$  for each  $\Delta F$  and for each  $x_0$ . We have  $\delta_2 J(x_0, \hat{F}; \Delta F) = x_0^\top \delta\varphi(\hat{F}; \Delta F)x_0$ , which implies that  $\delta\varphi(\hat{F}; \Delta F) = 0$  for all increments  $\Delta F$ . Hence

$$\partial\varphi(\hat{F}) = 0. \quad (4)$$

Next, we introduce the map  $\Phi : \mathcal{F} \times \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$  by

$$\Phi(F, P) = (A + BF)^\top P + P(A + BF) + Q + F^\top RF.$$

By definition, see (2), we have  $\Phi(F, \varphi(F)) = 0$  for all  $F \in \mathcal{F}$ . Taking the derivative of this equality and applying the chain rule yields

$$\partial_1 \Phi(F, \varphi(F)) + \partial_2 \Phi(F, \varphi(F)) \partial\varphi(F) = 0 \text{ for all } F \in \mathcal{F}.$$

Substituting  $F = \hat{F}$  in this equality, and using (4), we obtain  $\partial_1 \Phi(\hat{F}, \varphi(\hat{F})) = 0$ , or, equivalently,

$$\delta_1 \Phi(\hat{F}, \varphi(\hat{F}); \Delta F) = 0 \text{ for all } \Delta F. \quad (5)$$

The differential of  $\Phi$  with respect to its first argument with increment  $\Delta F$  is

$$\delta_1 \Phi(F, P; \Delta F) = \Delta F^\top (B^\top P + RF) + (PB + F^\top R) \Delta F.$$

Combining this result with (5) produces

$$\Delta F^\top (B^\top \varphi(\hat{F}) + R\hat{F}) + (\varphi(\hat{F})B + \hat{F}^\top R) \Delta F = 0 \text{ for all } \Delta F,$$

which clearly implies that  $B^\top \varphi(\hat{F}) + R\hat{F} = 0$ , or, equivalently,  $\hat{F} = -R^{-1}B^\top \varphi(\hat{F})$ . Now, since  $\Phi(\hat{F}, \varphi(\hat{F})) = 0$ , we conclude that  $X := \varphi(\hat{F})$  is the stabilizing solution of the ARE.  $\square$

Combining this theorem with the statement preceding the theorem we have the following result.

**Corollary 3.2** *The IRZILQ problem has a solution in the class of linear time-invariant state feedback controls (i.e. (iii) holds) if and only if the ARE (3) has a stabilizing solution. If this condition holds, the solution is uniquely given by  $F = -R^{-1}B^\top X$  where  $X$  is the stabilizing solution of the ARE.*

The existence of the stabilizing solution of the ARE can for instance be verified by checking whether the corresponding Hamiltonian matrix has no purely imaginary eigenvalues, and whether a rank condition on the matrix sign of a certain matrix is satisfied (Lancaster & Rodman, 1995, Theorem 22.4.1, or Laub, 1991, p. 175). An extensive literature on algorithms for accurately computing the matrix sign exists, and a comprehensive list of references can be found in the review paper of Laub (1991).

## 4 Stochastic Interpretation

In addition to the five statements (i), ..., (v) presented in Section 1, we consider a sixth statement in this section. This statement is related to the infinite-horizon LQG problem with the state available for feedback (see e.g. Anderson & Moore, 1989, Section 8.2). We will show that the sixth statement, which will formally be introduced below, is equivalent to statement (iii) (Lemma 4.2 below). This implies that the existence of the stabilizing solution of the ARE is also necessary and sufficient for the solvability of the IRSILQG problem (introduced below) in the class of linear internally stabilizing state feedback controls.

Let  $\mathcal{S}$  be the set of all real positive semi-definite  $n \times n$ -matrices. Consider the following indefinite, regular, full-state-information, infinite-horizon LQG (IRSILQG) problem. Find a feedback matrix  $\hat{F} \in \mathcal{F}$  which minimizes for each  $S \in \mathcal{S}$  the criterion  $L : \mathcal{S} \times \mathcal{F} \rightarrow \mathbb{R}$ , defined by

$$L(S, F) = \lim_{T \rightarrow \infty} \frac{1}{T} E \left( \int_0^T (x^\top Q x + u^\top R u) dt \right) \quad (6)$$

with  $Q = Q^\top$ ,  $R > 0$ ,  $u = Fx$ , and where  $x$  is generated by a linear noisy system, i.e.

$$\dot{x} = Ax + Bu + w \quad (7)$$

with a white Gaussian noise  $w$  of zero mean and covariance  $S\delta(t - \tau)$ . The initial state is assumed to be a Gaussian random variable independent of  $w$ . Our setting differs from the setup of Anderson & Moore (1989) in that we do not assume  $Q$  to be positive semi-definite and furthermore, we require the solution to be independent of the covariance matrix  $S$ . Indefinite LQG problems have recently been studied by several authors. See e.g. Ait Rami, Zhou, & Moore (2000) and references therein. However, the setting of Ait Rami et al. is different in several respects. In particular, they consider multiplicative noise.

We call the IRSILQG problem solvable if:

$$(vi) \quad \exists \hat{F} \in \mathcal{F} \quad \forall S \in \mathcal{S} \quad \forall F \in \mathcal{F} \quad L(S, \hat{F}) \leq L(S, F).$$

We show that (iii)  $\Leftrightarrow$  (vi) (Lemma 4.2). Using the result of the previous section this implies that (vi)  $\Leftrightarrow$  (v) (Corollary 4.3). The implication (v)  $\Leftrightarrow$  (vi) is well-known for a positive semi-definite  $Q$ , but is usually proved in a different way. We first need the following lemma.

**Lemma 4.1** *For each  $S \in \mathcal{S}$  and  $F \in \mathcal{F}$  we have*

$$L(S, F) = \text{tr}(S\varphi(F)), \tag{8}$$

where  $\varphi$  is defined in Section 2.

**Proof** Let  $S \in \mathcal{S}$  and  $F \in \mathcal{F}$ . Analogously to e.g. Anderson & Moore, 1989, Equation (8.2-11) one can show that  $L(S, F)$  can be written as  $L(S, F) = \text{tr}(W(Q + F^\top RF))$ , where  $W$  is the unique solution of the Lyapunov equation

$$(A + BF)W + W(A + BF)^\top = -S.$$

Denote  $P = \varphi(F)$ . Multiplying the Lyapunov equation (2) by  $W$  produces

$$W(Q + F^\top RF) = -W(A + BF)^\top P - WP(A + BF).$$

Hence, it is easily seen that  $L(S, F) = \text{tr}(SP)$ . □



**Lemma 4.2** *Let the cost functionals  $J$  and  $L$ , as defined in (1) and (6) respectively, correspond to the same parameter set  $(A, B, Q, R)$ . Then (iii) and (vi) are equivalent.*

**Proof** ( $\Rightarrow$ ) Choose  $\hat{F}$  as the feedback matrix corresponding to statement (iii). Let  $S \in \mathcal{S}$  and  $F \in \mathcal{F}$ . Since  $S$  is positive semi-definite there exists a matrix  $Y$  such that  $S = YY^\top$ . Denote the  $i$ -th column of  $Y$  by  $y_i$ . From (8) it follows that

$$L(S, F) = \text{tr}(S\varphi(F)) = \text{tr}(Y^\top \varphi(F)Y) = \sum_{i=1}^n y_i^\top \varphi(F)y_i = \sum_{i=1}^n J(y_i, F).$$

For each  $i = 1, \dots, n$  we have  $J(y_i, \hat{F}) \leq J(y_i, F)$ . Hence

$$L(S, \hat{F}) = \sum_{i=1}^n J(y_i, \hat{F}) \leq \sum_{i=1}^n J(y_i, F) = L(S, F).$$

( $\Leftarrow$ ) Choose  $\hat{F}$  as the feedback matrix corresponding to statement (vi). Let  $x_0 \in \mathbb{R}^n$  and  $F \in \mathcal{F}$ . Define the matrix  $S := x_0 x_0^\top$ . Clearly, we have  $S \in \mathcal{S}$ , which implies that  $L(S, \hat{F}) \leq L(S, F)$ . Hence, using (8) we find

$$J(x_0, \hat{F}) = x_0^\top \varphi(\hat{F})x_0 = \text{tr}(S\varphi(\hat{F})) = L(S, \hat{F}) \leq L(S, F) = J(x_0, F). \quad \square$$

Combining this result with the results of the previous section yields the following result.

**Corollary 4.3** *The IRSILQG problem has a solution (independent of the covariance matrix) if and only if the ARE (3) has a stabilizing solution. If this condition holds, the solution is uniquely given by  $F = -R^{-1}B^\top X$  with  $X$  the stabilizing solution of the ARE.*

## 5 An Application to LQ Differential Games

In this section we use the equivalence results from the two preceding sections to characterize feedback Nash equilibria in infinite-horizon LQ differential games both in a deterministic and a stochastic context.

The following notation will be used. For an  $N$ -tuple  $\hat{F} = (\hat{F}_1, \dots, \hat{F}_N) \in \Gamma_1 \times \dots \times \Gamma_N$  for given sets  $\Gamma_i$ , we shall write  $\hat{F}_{-i}(\alpha) = (\hat{F}_1, \dots, \hat{F}_{i-1}, \alpha, \hat{F}_{i+1}, \dots, \hat{F}_N)$  with  $\alpha \in \Gamma_i$ .

## 5.1 The Deterministic Case

Consider the cost function of player  $i$  defined by

$$J_i(x_0, F_1, \dots, F_N) = \int_0^\infty \left( x^\top Q_i x + \sum_{j=1}^N u_j^\top R_{ij} u_j \right) dt$$

with  $u_j = F_j x$  for  $j = 1, \dots, N$ , and where  $x$  is generated by

$$\dot{x} = Ax + \sum_{j=1}^N B_j u_j, \quad x(0) = x_0.$$

Assume that  $Q_i$  is symmetric,  $R_{ii}$  is positive definite and  $(F_1, \dots, F_N) \in \mathcal{F}_N$ , where

$$\mathcal{F}_N = \left\{ (F_1, \dots, F_N) \left| A + \sum_{j=1}^N B_j F_j \text{ is stable} \right. \right\}. \quad (9)$$

This last assumption spoils the rectangular structure of the strategy spaces, i.e. choices of feedback matrices cannot be made independently. However, such a restriction is motivated by the fact that closed-loop stability is usually a common objective.

In our setting the concept of a feedback Nash equilibrium is defined as follows.

**Definition 5.1** An  $N$ -tuple  $\hat{F} = (\hat{F}_1, \dots, \hat{F}_N) \in \mathcal{F}_N$  is called a *feedback Nash equilibrium* if for all  $i$  the following inequality holds:

$$J_i(x_0, \hat{F}) \leq J_i(x_0, \hat{F}_{-i}(\alpha))$$

for each  $x_0$  and for each state feedback matrix  $\alpha$  such that  $\hat{F}_{-i}(\alpha) \in \mathcal{F}_N$ .

Next, consider the set of coupled algebraic Riccati equations (see also Başar and Olsder, 1999, Proposition 6.8):

$$\begin{aligned} Q_i + A^\top X_i + X_i A - \sum_{j=1, j \neq i}^N X_i B_j R_{jj}^{-1} B_j^\top X_j - \sum_{j=1, j \neq i}^N X_j B_j R_{jj}^{-1} B_j^\top X_i + \\ - X_i B_i R_{ii}^{-1} B_i^\top X_i + \sum_{j=1, j \neq i}^N X_j B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j^\top X_j = 0, \quad i = 1, \dots, N. \end{aligned} \quad (10)$$

A *stabilizing solution* of (10) is an  $N$ -tuple  $(X_1, \dots, X_N)$  of real symmetric  $n \times n$  matrices satisfying (10) such that  $A - \sum_{j=1}^N B_j R_{jj}^{-1} B_j^\top X_j$  is stable. In contrast to the stabilizing solution

of (3), stabilizing solutions of (10) are not necessarily unique (see e.g. Weeren et al. (1999)). The next theorem states that feedback Nash equilibria are completely characterized by stabilizing solutions of (10). Its proof follows straightforwardly from Corollary 3.2.

**Theorem 5.2** *Let  $(X_1, \dots, X_N)$  be a stabilizing solution of (10) and define  $F_i := -R_{ii}^{-1} B_i^\top X_i$  for  $i = 1, \dots, N$ . Then  $(F_1, \dots, F_N)$  is a feedback Nash equilibrium. Conversely, if  $(F_1, \dots, F_N)$  is a feedback Nash equilibrium, there exists a stabilizing solution  $(X_1, \dots, X_N)$  of (10) such that  $F_i = -R_{ii}^{-1} B_i^\top X_i$ .*

The restriction that feedback matrices belong to the set  $\mathcal{F}_N$  is essential. Indeed, there exist feedback Nash equilibria in which a player can improve unilaterally by choosing a feedback matrix for which the closed-loop system is unstable (Mageirou, 1976).

## 5.2 The Stochastic Case

Now consider

$$L_i(S, F_1, \dots, F_N) = \lim_{T \rightarrow \infty} \frac{1}{T} E \left( \int_0^T \left( x^\top Q_i x + \sum_{j=1}^N u_j^\top R_{ij} u_j \right) dt \right)$$

with  $u_j = F_j x$  for  $j = 1, \dots, N$ , and where  $x$  is generated by a linear noisy system, i.e.

$$\dot{x} = Ax + \sum_{j=1}^N B_j u_j + w$$

with  $w, x_0$  as defined in (7) and  $Q_i, R_{ij}$  as defined in the previous section. The information structure of the players is assumed to be a perfect state feedback pattern. Next, introduce

**Definition 5.3** An  $N$ -tuple  $\hat{F} = (\hat{F}_1, \dots, \hat{F}_N) \in \mathcal{F}_N$  is called a *stochastic variance-independent feedback Nash equilibrium* if for all  $i$  the following inequality holds:

$$L_i(S, \hat{F}) \leq L_i(S, \hat{F}_{-i}(\alpha))$$

for each  $S \in \mathcal{S}$  and for each state feedback matrix  $\alpha$  such that  $\hat{F}_{-i}(\alpha) \in \mathcal{F}_N$ .

Then, the following theorem, which follows straightforwardly from Corollary 4.3, characterizes all stochastic variance-independent feedback Nash equilibria.

**Theorem 5.4** *Let  $(X_1, \dots, X_N)$  be a stabilizing solution of (10) and define  $F_i := -R_{ii}^{-1} B_i^\top X_i$  for  $i = 1, \dots, N$ . Then  $(F_1, \dots, F_N)$  is a stochastic variance-independent feedback Nash equilibrium. Conversely, if  $(F_1, \dots, F_N)$  is a stochastic variance-independent feedback Nash equilibrium, there exists a stabilizing solution  $(X_1, \dots, X_N)$  of (10) such that  $F_i = -R_{ii}^{-1} B_i^\top X_i$ .*

We conclude that for a given parameter set  $(A, B_i, Q_i, R_{ij})$  the set of stochastic variance-independent feedback Nash equilibria coincides with the set of feedback Nash equilibria corresponding to the deterministic case.

## 6 Concluding Remarks

We have shown that the existence of a stabilizing solution of the ARE is a necessary and sufficient condition for the unique solvability of the IRZILQ problem in the class of linear state feedback controls. Furthermore, we have also shown that this unique solution coincides with the unique solution of the IRSILQG problem in the class of linear state feedback controls. The equivalence results have been generalized to a differential game setting in both a deterministic and stochastic context, with a perfect state feedback information pattern for the players.

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